101. Substituting gives $4a - 3 = 17$, so $a = 5$.

102. Let $x = 0.\dot{4}\dot{2}$. Writing longhand,

$$\begin{aligned} x &= 0.424242... \\ 100x &= 42.424242... \end{aligned}$$

Subtracting, the decimal expansions cancel, giving $99x = 42$. Hence, $x = \frac{42}{99} = \frac{14}{33}$.

103. Since probabilities must sum to 1, we know that

$$\begin{aligned} & a + \left(\tfrac{1}{2} - a\right) + 3a + a = 1 \\ \Longrightarrow\ & 4a = \tfrac{1}{2} \\ \Longrightarrow\ & a = \tfrac{1}{8}. \end{aligned}$$

104. The squared distance from the origin to the point $(2, 3)$ is $2^2 + 3^2 = 13$. For points on the circle, that value is 10. So, the point lies outside the circle.

105. (a) The relevant index law is $\left(a^b\right)^c \equiv a^{bc}$.

(b) By definition, $\log_a b$ is "the power you need to raise a by to get b". So, using the result in part (a), each of the factors simplifies by definition:

$$\begin{aligned} p^{\log_p 3} \times \left(q^{\log_q 2}\right)^2 &= 3 \times 2^2 \\ &= 12. \end{aligned}$$

106. By Pythagoras, the squared magnitude is

$$\begin{aligned} \left(-\tfrac{7}{25}\right)^2 + \left(\tfrac{24}{25}\right)^2 &= \tfrac{7^2+24^2}{25^2} \\ &= 1. \end{aligned}$$

The magnitude of $\mathbf{r} = -\frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}$ is 1, so $\mathbf{r}$ is a unit vector, as required.

Nota Bene

$(7, 24, 25)$ is a Pythagorean triple. The same result holds for any Pythagorean triple. If $a^2 + b^2 = c^2$, then $\mathbf{r} = -\frac{a}{c}\mathbf{i} + \frac{b}{c}\mathbf{j}$ is a unit vector.

107. (a) A projectile is modelled as

- having negligible size, i.e. as a particle,
- which is acted on only by gravity, so that acceleration is $a = 9.8$ ms^{-2} downwards.

(b) At landing, the vertical displacement is zero. Substituting $s = 0$, $u = 19.6$ and $a = -9.8$, we get $0 = 19.6t - \frac{1}{2}gt^2$. This is a quadratic, with roots $t = 0$ (take-off) and

$$t = \frac{2 \cdot 19.6}{g} = 4 \text{ (landing)}.$$

So, the time of flight is 4 seconds.

(c) In 4 seconds, the horizontal range is given by $d = 12.5 \times 4 = 50$ metres.

108. The polynomial differentiation formula is

$$\begin{aligned} & y = x^n \\ \Longrightarrow\ & \frac{dy}{dx} = nx^{n-1}. \end{aligned}$$

Using this,

$$\begin{aligned} & y = x^2 \\ \Longrightarrow\ & \frac{dy}{dx} = 2x. \end{aligned}$$

Evaluating at $x = 1$ gives a gradient of 2. The relevant point is $(x_1, y_1) = (1, 1)$. The equation of a line with gradient m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

Substituting in, the equation of the tangent is

$$\begin{aligned} & y - 1 = 2(x - 1) \\ \Longrightarrow\ & y = 2x - 1, \text{ as required.} \end{aligned}$$

109. Writing over base 2, $8^x \equiv \left(2^3\right)^x \equiv \left(2^x\right)^3 = y^3$.

Nota Bene

Swapping the indices in the above is an instance of the index law

$$\left(a^b\right)^c \equiv a^{bc}.$$

It is a direct result of this law that the order in which powers are applied doesn't matter:

$$\left(a^b\right)^c \equiv a^{bc} \equiv a^{cb} \equiv \left(a^c\right)^b.$$

110. The percentage error in an estimation of a quantity s is given by

$$\frac{s_{\text{estimated}} - s_{\text{actual}}}{s_{\text{actual}}}.$$

So, the percentage errors are

(a) $\dfrac{0.1 - \sin 0.1}{\sin 0.1} = 0.167\%$ (3sf),

(b) $\dfrac{0.5 - \sin 0.5}{\sin 0.5} = 4.29\%$ (3sf).

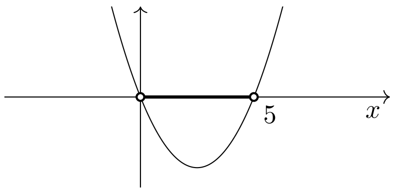
Nota Bene

The small-angle approximations, for θ in radians, which become increasingly accurate in the limit as $\theta \to 0$, are

$$\begin{aligned} \sin\theta &\approx \theta, \\ \cos\theta &\approx 1 - \tfrac{1}{2}\theta^2, \\ \tan\theta &\approx \theta. \end{aligned}$$

111. Since $(2x + a)$ and $(x + a)$ are factors, we know, by the factor theorem, that $x = -\frac{a}{2}$ and $x = -a$ are roots. These must be 3 and 6 respectively, so $a = -6$.

112. The boundary equation is $x^2 - 5x = 0$. This has roots at $x = 0$ and $x = 5$. So, the parabola $y = x^2 - 5x$ has x intercepts at $x = 0$ and $x = 5$:



Since $y = x^2 - 5x$ is a positive parabola, we need x between and not including the roots, as shown above. The solution is $x \in (0, 5)$.

113. The equation of motion is

$$\begin{aligned} &k^2 - 1 - (k+1) = 10 \\ \Longrightarrow\ &k^2 - k - 12 = 0 \\ \Longrightarrow\ &(k-4)(k+3) = 0 \\ \Longrightarrow\ &k = -3, 4. \end{aligned}$$

The positive value of k is 4.

114. The rule for integration is

$$\int kx^n \, dx = \frac{k}{n+1} x^{n+1} + c.$$

Applying this term by term,

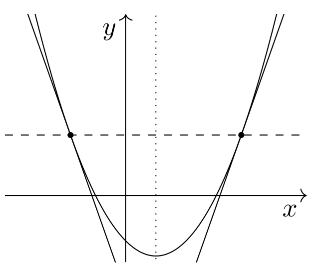
$$\int 6x^2 + 5 \, dx = 2x^3 + 5x + c.$$

— Nota Bene —

Since integration is the reverse of differentiation, integral results are best understood by taking the answer and differentiating it. In this case,

$$\frac{d}{dx}(2x^3 + 5x + c) = 6x^2 + 5.$$

115. A parabola of the form $y = ax^2 + bx + c$ has a line of symmetry parallel to the y axis, shown dotted below. Two points with the same y value $y = k$ must be reflections of each other in this line.



The tangents at these points are reflections of each other, so their gradients are negatives. In other words $m_1 = -m_2$, which gives $m_1 + m_2 = 0$.

116. The prime factorisation of 28 is $28 = 2^2 \cdot 7$. So, the relevant divisors are $\{1, 2, 4, 7, 14\}$. This gives $1 + 2 + 4 + 7 + 14 = 28$. Hence, 28 is a perfect number, as required.

117. In radians, the sum of the interior angles of an n-gon is given by $\pi(n-2)$. For a pentagon, $n = 5$, so the sum is 3π. This gives

$$\begin{aligned} &\tfrac{4\pi}{10} + \tfrac{5\pi}{10} + \tfrac{6\pi}{10} + \tfrac{7\pi}{10} + \theta = 3\pi \\ \Longrightarrow\ &\theta = \tfrac{4\pi}{5}. \end{aligned}$$

118. (a) The implication does not hold, because every positive number has two real square roots. So, $(-2, 4)$ is a counterexample, satisfying $y = x^2$ but not $x = y^{\frac{1}{2}}$.

(b) The implication does hold, because every real number has exactly one real cube root.

— Nota Bene —

The expressions $x^{\frac{1}{2}}$ and $\sqrt{x}$, which are identical to one another, denote the *positive* square root of x. This is why, when solving e.g. $x^2 = 7$, you must write in a $\pm$ sign next to the square root symbol:

$$\begin{aligned} &x^2 = 7 \\ \Longleftrightarrow\ &x = \pm\sqrt{7}. \end{aligned}$$

119. The intersection of the first two lines is at $(2, -1)$. If the third line is to pass through this point, then $2 \times 2 - (-1) = k$, so $k = 5$.

— Nota Bene —

In the Latin, con-current literally means *running together*. Much mathematical terminology has a classical basis, because the ideas were formalised when Latin was still the lingua franca.

120. Factorising the quadratic,

$$\begin{aligned} &\left(3^x\right)^2 - 6 \cdot \left(3^x\right) - 27 = 0 \\ \Longrightarrow\ &\left(3^x + 3\right)\left(3^x - 9\right) = 0 \\ \Longrightarrow\ &3^x = -3 \text{ or } 9. \end{aligned}$$

The first equation $3^x = -3$ has no roots, as 3^x is always positive. The second equation gives $x = 2$.

— Alternative Method —

Let $z = 3^x$. The equation is now

$$\begin{aligned} &z^2 - 6z - 27 = 0 \\ \Longrightarrow\ &(z+3)(z-9) = 0 \\ \Longrightarrow\ &z = -3, 9. \end{aligned}$$

This gives $3^x = -3, 9 \implies x = 2$.

121. For intersections,

$$\begin{aligned} & 5x^2 - 20x + 40 = 2x^2 + 15x + 52 \\ \Longrightarrow\ & 3x^2 - 35x - 12 = 0 \\ \Longrightarrow\ & (x-12)(3x+1) = 0 \\ \Longrightarrow\ & x = 12, -\tfrac{1}{3}. \end{aligned}$$

Evaluating y at these x values, the (x, y) solution points are $(12, 520)$ and $\left(-1/3, 425/9\right)$.

— Nota Bene —

The algebraic operation above is often described as "making them equal to each other" or something similar. You're better off thinking of the operation as "substitution for y", a more rigorous idea which carries over effectively into harder problems.

122. We can assume that $b \neq 0$, since otherwise the LHS would be undefined. Putting the LHS over a common denominator,

$$\begin{aligned} & \frac{1}{b(ab-1)} + \frac{1}{b} \\ \equiv\ & \frac{1}{b(ab-1)} + \frac{ab-1}{b(ab-1)} \\ \equiv\ & \frac{1+ab-1}{b(ab-1)} \\ \equiv\ & \frac{ab}{b(ab-1)} \\ \equiv\ & \frac{a}{ab-1}. \end{aligned}$$

This proves the result.

— Nota Bene —

The best technique for proving identities is almost always as above: take one side, as an expression, and manipulate it to reach the other. As a rule of thumb, start with the *more complicated* side. This is because simplification is much more likely to lead you in the right direction than the non-existent technique of "complicatification".

123. (a) Multiplying out and simplifying yields

$$\begin{aligned} & x^2 + y^2 = (x-1)^2 + (y-1)^2 \\ \Longleftrightarrow\ & x^2 + y^2 = x^2 - 2x + 1 + y^2 - 2y + 1 \\ \Longleftrightarrow\ & 0 = -2x - 2y + 2 \\ \Longleftrightarrow\ & x + y = 1. \end{aligned}$$

This is the equation of a straight line.

(b) Each side of the equation is Pythagorean: an expression for a squared distance. The LHS is the squared distance between (x, y) and $(0, 0)$; the RHS is the squared distance between (x, y) and $(1, 1)$. So, the points which satisfy the equation are equidistant from $(0, 0)$ and $(1, 1)$. This gives the perpendicular bisector.

124. Dividing through by x gives $\frac{y}{x} = ax + b$. This is a linear relationship between $\frac{y}{x}$ and x.

— Nota Bene —

The above is part of a method for testing whether data fits a quadratic relationship. If it is suspected that (x, y) data follows $y = ax^2 + bx$, then a plot of y/x against x should produce a straight line.

125. From the perimeter, $2(x + y) = 22$, which is $x + y = 11$. From the area, $xy = 28$. Substituting for y,

$$\begin{aligned} & x(11-x) = 28 \\ \Longrightarrow\ & x^2 - 11x + 28 = 0 \\ \Longrightarrow\ & x = 4, 7. \end{aligned}$$

So, the side lengths are 4 cm and 7 cm.

— Nota Bene —

In a problem like this, in which the numbers work nicely, it may be possible to write the answer down simply by looking at it. However, we are looking for rigorous methods that (a) *guarantee* that the solution obtained is the full solution, and (b) carry through to tougher problems.

126. With replacement, the probability is

$$p = 1 \times \tfrac{4}{52} \times \tfrac{4}{52} \times \tfrac{4}{52}.$$

Without replacement, the probability is

$$p = 1 \times \tfrac{3}{51} \times \tfrac{2}{50} \times \tfrac{1}{49}.$$

Comparing these, $4/52 > 3/51$, and similarly for the other fractions. So, four of a kind is more probable with replacement.

127. This is a difference of two squares $(p+q)(p-q)$, where $p = x - 2$ and $q = \sqrt{11}$. Multiplying out,

$$\begin{aligned} & \left(x - 2 + \sqrt{11}\right)\left(x - 2 - \sqrt{11}\right) \\ \equiv\ & (x-2)^2 - 11 \\ \equiv\ & x^2 - 4x - 7. \end{aligned}$$

This is a quadratic with integer coefficients.

128. (a) Firstly, we rearrange:

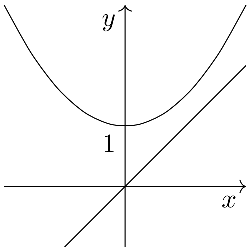
$$\begin{aligned} & \tfrac{1}{2}x^2 + 1 = x \\ \Longrightarrow\ & x^2 + 2 = 2x \\ \Longrightarrow\ & x^2 - 2x + 2 = 0. \end{aligned}$$

This is a quadratic with discriminant

$$\Delta = b^2 - 4ac = 4 - 4 \cdot 2 = -4.$$

Since $\Delta < 0$, the equation has no real roots.

(b) We know that $y = \frac{1}{2}x^2 + 1$ is a positive parabola, symmetrical about the y axis, with an intercept at $y = 1$. Furthermore, from part (a), we know that it does not intersect $y = x$. So, the sketch is



(c) Since the parabola is always above the line $y = x$, its y values are always greater than its x values. So, the outputs of the function f are always greater than its inputs.

129. Two odd numbers can be written as $2a + 1$ and $2b + 1$, where $a, b \in \mathbb{Z}$. Their product is

$$\begin{aligned} &(2a+1)(2b+1) \\ \equiv\ &4ab + 2a + 2b + 1 \\ \equiv\ &2(2ab + a + b) + 1. \end{aligned}$$

Since $(2ab + a + b)$ is an integer, $2(2ab + a + b)$ is even, so $2(2ab + a + b) + 1$ must be odd. QED.

130. Since $(1, b)$ is on both lines, we know that

$$\begin{aligned} &a + b = 1, \\ &11 - 2 + 3b = 0. \end{aligned}$$

The second equation gives us $b = -3$, then the first gives us $a = 4$.

131. Differentiating both sides,

$$\begin{aligned} &3y - 1 = 2x^{\frac{3}{2}} \\ \Longrightarrow\ &3\frac{dy}{dx} = 2 \cdot \tfrac{3}{2}x^{\frac{1}{2}} \\ \Longrightarrow\ &\frac{dy}{dx} = x^{\frac{1}{2}} \\ \Longrightarrow\ &\frac{dy}{dx} = \sqrt{x}. \end{aligned}$$

— Nota Bene —

You can, if you want to, rearrange to make y the subject before performing the differentiation. But, since differentiation is an algebraic operation like any other, there's no *need* to rearrange. If the two sides of an equation are equal to each other, then the derivatives of those two sides must be equal.

132. By the factor theorem, since $(x + 2)$ is a factor, $x = -2$ is a root. Substituting this value in,

$$\begin{aligned} &(-2)^2 + 5(-2) + k = 0 \\ \Longrightarrow\ &k = 6. \end{aligned}$$

133. A fraction can only be zero when its numerator is zero, so the roots are $x = \pm 1$.

134. (a) This is false: $x = -1$ is a counterexample, for which $x^2 = 1$ but $x^5 \neq 1$.

(b) This is true: taking the fifth root,

$$\begin{aligned} x^5 = 1 &\Longleftrightarrow x = 1 \\ &\Longrightarrow x^2 = 1. \end{aligned}$$

135. The boundary equation is $3x^2 + 6x + 1 = 0$. The quadratic formula gives $x \approx -0.184, -1.816$. We require the positive quadratic $3x^2 + 6x + 1$ to be less than zero, which is true of all values between the roots of the boundary equation. The solution (to 3dp) of the inequality is

$$\begin{aligned} &3x^2 + 6x + 1 < 0 \\ \Longrightarrow\ &x \in (-1.816, -0.184). \end{aligned}$$

The only integer in this interval is $x = -1$.

136. Each tile can be placed independently in one of two orientations, so the possibility space consists of $2^4 = 16$ outcomes.

(a) Only one outcome produces an (approximate) square, so the probability is $\frac{1}{16}$.

(b) There are two outcomes, corresponding to the two orientations, in which the four stripes are parallel. So, the probability is $\frac{2}{16} = \frac{1}{8}$.

137. Differentiating term by term,

$$\begin{aligned} &y = x^5 - 4x^3 + 10x^2 + 5 \\ \Longrightarrow\ &\frac{dy}{dx} = 5x^4 - 12x^2 + 20x. \end{aligned}$$

Evaluating this at $x = 2$ gives $\frac{dy}{dx} = 72$.

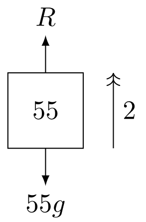
138. There are 2 ways of placing the heavy books. Then the remaining eight books can go in any order in 8 positions, which means 8! possible orders. So the total number of ways is $2 \times 8! = 80640$.

139. The second statement implies the first. And $x = 2$ is a counterexample to the forwards implication. So, the statements should be linked by

$$(x-1)(x-2) = 0 \Longleftarrow x = 1.$$

140. Distance is unaffected by translation of the entire problem. Translating by $-a$ in the x direction and $-b$ in the y direction, the distance is the same as that between $(0, 0)$ and $(3, -4)$. By Pythagoras, this is $\sqrt{3^2 + 4^2} = 5$.

141. (a) The force diagram for the woman is



(b) Using NII,

$$R - 55g = 55 \cdot 2$$
$$\implies R = 55(2 + g) = 649.$$

So, the force is 649 N.

(c) The forces in part (b) and this part form a Newton pair, describing the single interaction between the woman's feet and the lift floor. Hence, the force is 649 N by definition.

142. The longer diagonal divides the kite into a pair of congruent triangles, each with base 8 and height 3. So, the overall area is

$$A = 2 \times \tfrac{1}{2} \cdot 8 \cdot 3$$
$$= 24.$$

143. We get two equations by substituting: $a - 5 = 0$ and $b^2 + ab - 6$. The first tells us that $a = 5$, and the second that $b = 1$ or -6. But since $x = 1$ and $x = b$ are distinct, b must be -6.

144. Since the lily doubles in size every day and will be the size of the pond on day 30, it will be half the size of the pond on day 29.

145. Multiplying up by $(x-1)^2$,

$$\frac{x+1}{(x-1)^2} = 1$$
$$\implies x + 1 = (x-1)^2$$
$$\implies x + 1 = x^2 - 2x + 1$$
$$\implies 0 = x^2 - 3x$$
$$\implies 0 = x(x-3)$$
$$\implies x = 0, 3.$$

146. (a) Evaluating the quadratic at $x = \frac{1}{2}$,

$$16x^2 - 44x + 6 \Big|_{x=\frac{1}{2}} = -12.$$

(b) In part (a), $-12 \neq 0$, so $x = \frac{1}{2}$ is not a root of the quadratic. By the factor theorem, then, $(2x - 1)$ cannot be a factor.

147. Rearranging to the usual quadratic form,

$$100x^2 - 41x + 5 = 0.$$

The discriminant is $\Delta = 41^2 - 4 \cdot 5 \cdot 100 = -319$. This is negative, so there are no real roots.

148. These are a set of standard differentiation results, phrased in functional notation:

(a) $\mathrm{f}(x) = 0 \implies \mathrm{f}'(x) = 0$,

(b) $\mathrm{f}(x) = 1 \implies \mathrm{f}'(x) = 0$,

(c) $\mathrm{f}(x) = 1 + x \implies \mathrm{f}'(x) = 1$.

— NOTA BENE —

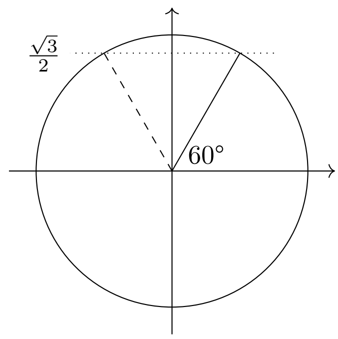
Written in Leibniz notation, these are

(a) $y = 0 \implies \frac{dy}{dx} = 0$,

(b) $y = 1 \implies \frac{dy}{dx} = 0$,

(c) $y = 1 + x \implies \frac{dy}{dx} = 1$.

149. Rearranging, $\sin\theta = \sqrt{3}/2$. The primary value is $\arcsin \sqrt{3}/2 = 60°$. Consulting the unit circle, the other angle which produces the same sine (y) value is at the dashed radius shown below.



Angles are given anticlockwise from the positive x axis. So, the second angle is 120°. The solution set, therefore, is $\{60°, 120°\}$.

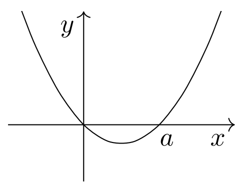
150. Since the parabola passes through $(-3, 0)$ and $(2, 0)$, it must be of the form

$$y = a(x+3)(x-2).$$

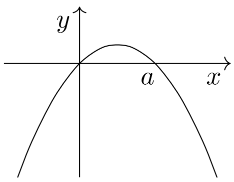
Substituting $(0, -12)$ gives $-12 = a \cdot -6$, so $a = 2$. Multiplying out, the equation of the parabola is $y = 2x^2 + 2x - 12$.

151. Angles in a triangle add up to π radians. So $1.42 + 0.46 + \theta = \pi$. Hence, $\theta = 1.26$ (2dp).

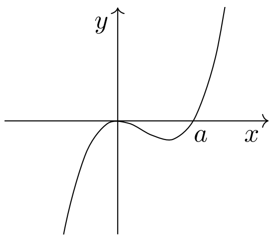
152. (a) $y = x(x - a)$ is a positive parabola passing through $x = 0$ and $x = a$, so



(b) $y = x(a - x)$ is as above, but reflected in the x axis, since $(a - x)$ is the negative of $(x - a)$.



(c) $y = x^2(x-a)$ is a cubic with a double root (just touches) at $x = 0$ and a single root (crosses) at $x = a$:



153. We rearrange to $|10 - 3x| = 4$. Since the absolute value of $10 - 3x$ is equal to 4, we know that

$$\begin{aligned} & 10 - 3x = \pm 4 \\ \Longrightarrow\ & -3x = \pm 4 - 10 \\ \Longrightarrow\ & -3x = -6, -14 \\ \Longrightarrow\ & x = 2, \tfrac{14}{3}. \end{aligned}$$

154. Defining x and y to be the dimensions of the lawn, we have $xy = 48$ and $x^2 + y^2 = 100$. Substituting the former into the latter yields

$$\begin{aligned} & x^2 + \frac{48^2}{x^2} = 100 \\ \Longrightarrow\ & x^4 - 100x^2 + 2304 = 0. \end{aligned}$$

This is a quadratic in x^2, with solution $x = \pm 8, \pm 6$. So, the perimeter is $6 + 6 + 8 + 8 = 28$ metres.

155. Since a, b, c, d is an AP, we know that $b = a + k$, $c = a + 2k$ and $d = a + 3k$, where k is the common difference. Evaluating $a + d$ and $b + c$,

$$\begin{aligned} a + d &= 2a + 3k, \\ b + c &= 2a + 3k. \end{aligned}$$

Quod erat demonstrandum (QED).

156. (a) The intervals overlap and stretch to infinity in both directions, so their union is the set of all real numbers $\mathbb{R}$.

(b) Here, we are looking for numbers which are in neither $(-\infty, -1]$ nor $[1, \infty)$. This is $(-1, 1)$.

— Nota Bene —

It is convention to give ∞ a round bracket. This is because ∞ is not a number. What we want to say is that e.g. $x \geq 1$ contains all real numbers greater than or equal to 1 and up to but not including ∞. Hence, the notation is $[1, \infty)$.

157. Using Pythagoras, the squared distances are

		d^2
$A : (3, 0)$	$B : (4, -2)$	5
$A : (3, 0)$	$C : (1, -3)$	13
$B : (4, -2)$	$C : (1, -3)$	10

So, points A and B are closest.

158. There are $4! = 24$ possible orders of four distinct objects, which are equally likely. Only one of these is successful, so the probability is $\frac{1}{24}$.

159. A quadratic function is symmetrical. Since this quadratic function has roots at 0 and 5, its vertex must therefore be at $\frac{5}{2}$. Substituting this in gives

$$x(x - 5)\Big|_{x=\frac{5}{2}} = -\tfrac{25}{4}.$$

— Alternative Method —

Multiplying out and differentiating,

$$\frac{dy}{dx} = 2x - 5.$$

Setting the derivative to zero for SPs, we get $x = \frac{5}{2}$. Evaluating at this x value gives $\left(\frac{5}{2}\right)\left(-\frac{5}{2}\right) = -\frac{25}{4}$.

160. This quartic has a common factor of x^2:

$$\begin{aligned} & 3x^4 - 14x^3 + 8x^2 = 0 \\ \Longrightarrow\ & x^2(3x^2 - 14x + 8) = 0 \\ \Longrightarrow\ & x^2(3x - 2)(x - 4) = 0 \\ \Longrightarrow\ & x = 0, \tfrac{2}{3}, 4. \end{aligned}$$

161. The endpoints of the chord are $(-1, 1)$ and $(2, 4)$. The gradient is $m = 1$. So, the equation of the curve is $y - 1 = 1(x + 1)$, which is $y = x + 2$. This crosses the y axis at $y = 2$, as required.

162. Solving $\mathrm{f}(x) = 100$ gives $x = 20$, and solving $\mathrm{g}(x) = 100$ gives $x = 16\frac{2}{3}$. So, as x increases from zero, $\mathrm{g}(x)$ reaches 100 first.

163. (a) Writing in terms of $\sqrt{2}$,

$$\begin{aligned} & \sqrt{2} + \sqrt{8} + \sqrt{32} \\ =\ & \sqrt{2} + 2\sqrt{2} + 4\sqrt{2} \\ =\ & 7\sqrt{2}. \end{aligned}$$

(b) Writing over a common denominator,

$$\begin{aligned} & \frac{1}{1 - \sqrt{a}} + \frac{1}{1 + \sqrt{a}} \\ \equiv\ & \frac{1 + \sqrt{a}}{1 - a} + \frac{1 - \sqrt{a}}{1 - a} \\ \equiv\ & \frac{2}{1 - a}. \end{aligned}$$

164. The factorial definition of ${}^n\mathrm{C}_r$ is

$$ {}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}. $$

So, the equation is

$$ \begin{aligned} & \frac{n!}{2(n-2)!} = 15 \\ \Longrightarrow\ & \frac{n(n-1)}{2} = 15 \\ \Longrightarrow\ & n^2 - n - 30 = 0 \\ \Longrightarrow\ & n = -5, 6. \end{aligned} $$

Since ${}^n\mathrm{C}_r$ is not defined for negative n, the only possible value of n is 6.

165. The process of finding all possible functions with a given derivative is integration:

(a) $\mathrm{f}(x) = \displaystyle\int 0\, dx = c$,

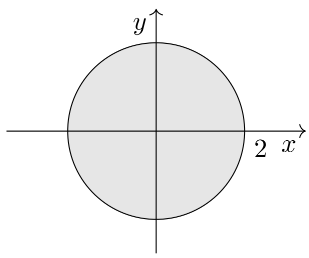
(b) $\mathrm{f}(x) = \displaystyle\int 3\, dx = 3x + c$,

(c) $\mathrm{f}(x) = \displaystyle\int 2x + 1\, dx = x^2 + x + c$.

166. Multiplying up to get rid of the fractions:

$$ \begin{aligned} & \frac{x}{x+1} - \frac{x-1}{x} = \frac{1}{2} \\ \Longrightarrow\ & 2x^2 - 2(x-1)(x+1) = x(x+1) \\ \Longrightarrow\ & x^2 + x - 2 = 0 \\ \Longrightarrow\ & x = -2, 1. \end{aligned} $$

167. The boundary equation is a circle with radius 2, centred at the origin. So the region we require is all points inside and on the circle:



168. We substitute $t = 0, 1$ to find the endpoints of this line segment. These are are $(0, 2)$ and $(2, -1)$, which lie at distances 2 and $\sqrt{5}$ from the origin. Since $\sqrt{5} > 2$, the greatest distance between a point on the line segment and the origin is $\sqrt{5}$. Hence, the line segment is never further than $\sqrt{5}$ from the origin.

169. The resultant force was zero. So, the magnitude of the resultant of the 4 and 6 N forces must be 2 N. When the 2 N force is removed, this becomes the resultant force for NII. Hence, the acceleration is $a = \frac{F}{m} = \frac{2}{10} = 0.2$ ms^{-2}.

170. Multiplying top and bottom of the fraction by the conjugate of the bottom,

$$ \begin{aligned} & \frac{1}{5 - 2\sqrt{6}} \\ = & \frac{\left(5 + 2\sqrt{6}\right)}{\left(5 - 2\sqrt{6}\right)\left(5 + 2\sqrt{6}\right)} \\ = & \frac{5 + 2\sqrt{6}}{25 - 24} \\ = & \, 5 + 2\sqrt{6}. \end{aligned} $$

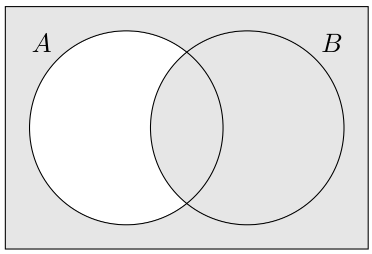
171. (a) Using $4 = 2^2$, we have $4^x = \left(2^2\right)^x$. Switching the order of the indices, this is $4^x \equiv (2^x)^2$.

(b) The equation is a quadratic in 2^x:

$$ \begin{aligned} & (2^x)^2 + 2^x - 6 = 0 \\ \Longrightarrow\ & (2^x + 3)(2^x - 2) = 0. \end{aligned} $$

(c) Using the factor theorem, $2^x = -3, 2$. Since 2^x is always positive, $2^x = -3$ has no roots. This leaves $2^x = 2$, which gives $x = 1$.

172. On the Venn diagram below, the set $A' \cup B$ is shaded and the set $A \cap B'$ is unshaded.



The sets are complements of each other, which means that their union is the universal set and their intersection is empty. This latter fact is what it means for them to be mutually exclusive.

——— Alternative Method ———

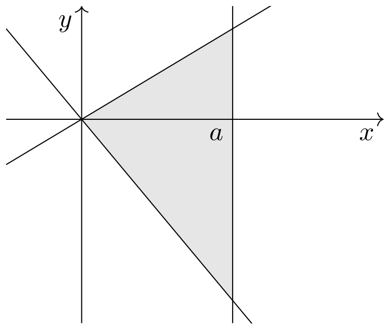
Assume, for a contradiction, that x is an element of both named sets. Since $x \in A \cap B'$, we know that $x \in B'$. So, since $x \in A' \cup B$, x must be an element of A'. But this means that x cannot be an element of $A \cap B'$. This is a contradiction. Hence, the two sets are mutually exclusive.

173. The distance, in the y direction, between the two lines is given by $2x - (-4x)$, which is $6x$. So, we can express the area as a single integral, adding up the values of this height from $x = 0$ to $x = a$. Algebraically, this is

$$ A = \int_0^a 6x\, dx = \Big[3x^2\Big]_0^a = 3a^2. $$

— Alternative Method —

The area enclosed is a triangle:



The y base of the triangle has length $6a$ and the x height is a. So, the triangle has area $\frac{1}{2} \cdot 6a \cdot a \equiv 3a^2$.

174. (a) Evaluating at $x = 2$,

$$f(2) = 2^3 + 6 \cdot 2^2 - 2 - 30 = 0.$$

Since $x = 2$ is a root of $f(x)$, $(x-2)$ is a factor.

(b) We take out the factor of $(x - 2)$ as follows. The first term in the quadratic factor must be x^2, so

$$\begin{aligned} &x^3 + 6x^2 - x - 30 \\ \equiv\ &(x-2)(x^2 + ...). \end{aligned}$$

The by-product is $-2x^2$. We want $6x^2$. So, we need $8x^2$ from the term in x. This gives

$$\begin{aligned} &x^3 + 6x^2 - x - 30 \\ \equiv\ &(x-2)(x^2 + 8x^2 + ...). \end{aligned}$$

The by-product is $-16x$. We want $-x$. So, we need $-15x$ from the constant term. This gives

$$\begin{aligned} &x^3 + 6x^2 - x - 30 \\ \equiv\ &(x-2)(x^2 + 8x^2 + 15). \end{aligned}$$

Factorising the quadratic,

$$f(x) = (x-2)(x+3)(x+5).$$

(c) By the factor theorem, $x = 2, -3, -5$.

— Alternative Method —

The factor of $(x - 2)$ can also be taken out by polynomial long division, which explicitly notates the process above. I don't personally use it, as the above is always quicker, but it does have the benefit of keeping a record of the process:

$$\begin{array}{r|l} & x^2 + 8x + 15 \\ \hline x-2 & x^3 + 6x^2 - x - 30 \\ & -x^3 + 2x^2 \\ \hline & 8x^2 - x \\ & -8x^2 + 16x \\ \hline & 15x - 30 \\ & -15x + 30 \\ \hline & 0 \end{array}$$

175. The following pairs of primes verify the conjecture up to $n = 20$:

$$\begin{aligned} 4 &= 2+2 & 14 &= 7+7 \\ 6 &= 3+3 & 16 &= 5+11 \\ 8 &= 3+5 & 18 &= 7+11 \\ 10 &= 3+7 & 20 &= 7+13 \\ 12 &= 5+7 \end{aligned}$$

176. The gradients of these lines are $m_1 = -a/b$ and $m_2 = b/a$. Multiplying these gives $m_1 m_2 = -1$, so the lines are perpendicular, as required.

177. A linear function can be expressed as $f(x) = ax+b$, for some constants a and b. Since $f'(1) = 2$, we know that $a = 2$. Then, substituting $x = 1$ gives $2 = 2 + b$, so $b = 0$. Hence $f(6) = 2 \cdot 6 = 12$.

— Nota Bene —

It is unusual to talk, in the manner of this question, about the derivative of a linear function, because you don't need calculus to analyse straight lines. Nevertheless, it is perfectly logical to do so: for linear f, the derivative $f'(x)$ is constant.

178. The odd integers form an AP. For $1, 3, 5, ...$, we have $a = 1$ and $d = 2$. So, the formula gives

$$\begin{aligned} S_n &= \tfrac{1}{2}n\,(2 + 2(n-1)) \\ &\equiv \tfrac{1}{2}n\,(2n) \\ &\equiv n^2. \end{aligned}$$

— Alternative Method —

The sum of the first n integers is

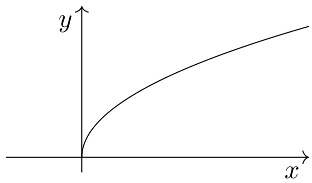
$$S_n = \tfrac{1}{2}n(n+1).$$

So, the sum of the first $2n$ integers is

$$\begin{aligned} S_{2n} &= \tfrac{1}{2}(2n)(2n+1) \\ &\equiv 2n^2 + n. \end{aligned}$$

The sum of the first n even integers is $2S_n$, which is $n(n+1) \equiv n^2 + n$. So, the sum of the first n odd integers is $2n^2 + n - (n+1) \equiv n^2$.

179. The square root graph $y = \sqrt{x}$ is the upper half $(y \geq 0)$ of the parabola $x = y^2$:



180. The possibility space is the same in both cases, consisting of $6^2 = 36$ equally likely outcomes. The numbers of successful outcomes are one $(6,6)$ and two $\{(5,6),(6,5)\}$ respectively. So, a five and a six is more likely than (twice as likely as) two sixes.

181. Solving for intersections,

$$\begin{aligned} & 2x - 1 = x^2 - 2x + 3 \\ \Longrightarrow\ & x^2 - 4x + 4 = 0 \\ \Longrightarrow\ & (x-2)^2 = 0 \\ \Longrightarrow\ & x = 2. \end{aligned}$$

There is only one point of intersection between the curve and the line. Hence, since the curve is a parabola, the point of intersection must be a point of tangency: the line does not cross the curve.

— Alternative Method —

The equation for intersections is

$$x^2 - 4x + 4 = 0.$$

This has discriminant $\Delta = 4^2 - 4 \cdot 4 = 0$. With $\Delta = 0$, we have exactly one point of intersection. Hence, as before, the line does not cross the curve.

182. (a) True.
(b) False, as $(-1)^3 \neq 1$.
(c) True.

183. (a) Completing the square for both x and y gives

$$\left(x + \tfrac{1}{2}\right)^2 + \left(y + \tfrac{1}{2}\right)^2 = \tfrac{1}{2}.$$

So, the centre is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$; the radius is $\frac{1}{\sqrt{2}}$.

(b) The circle C_1 passes through O, which is on $y = -x$. And both the circle C_1 and the line $y = -x$ are symmetrical in the line $y = x$. This symmetry dictates that the circle must be tangent to $y = -x$.

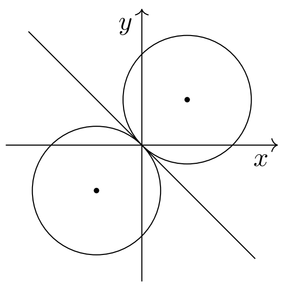
(c) The new circle has centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the same radius, so its equation is

$$\left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{2}.$$

This could also be expressed as

$$x^2 - x + y^2 - y = 0.$$

(d) The circles are both tangent to $y = -x$, on opposite sides of it:



184. To rationalise the denominator, we multiply top and bottom by its conjugate $4 + \sqrt{8}$:

$$\begin{aligned} & \frac{2\sqrt{2}}{4 - \sqrt{8}} \\ = & \frac{2\sqrt{2}(4 + \sqrt{8})}{16 - 8} \\ = & \frac{8\sqrt{2} + 8}{16 - 8} \\ = & \sqrt{2} + 1. \end{aligned}$$

185. (a) The object is in equilibrium. So, the resultant force is zero: the three forces, drawn tip-to-tail as vectors, must form a closed triangle. This is known as a triangle of forces.

(b) Since $45^2 + 60^2 = 75^2$, the $\triangle$ is right-angled. The angle between the shorter sides is $90°$.

186. (a) The equation of the parabola gives the points of intersection as $(5, -2)$ and $(10, 3)$. Subbing these into $y = mx + c$ gives $-2 = 5m + c$ and $3 = 10m + c$.

(b) Eliminating c, we get $m = 1$. Then $c = -7$. So, the equation of the line is $y = x - 7$.

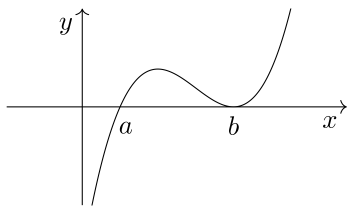
187. (a) Multiplying out before differentiating,

$$\begin{aligned} & y = x^2 - 3x + 2 \\ \Longrightarrow\ & \frac{dy}{dx} = 2x - 3. \end{aligned}$$

(b) Again multiplying out,

$$\begin{aligned} & y = x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} + 2 \\ \Longrightarrow\ & \frac{dy}{dx} = \tfrac{3}{2}x^{\frac{1}{2}} - 1 + x^{-\frac{1}{2}}. \end{aligned}$$

188. The cubic has a single root at $x = a$ and a double root at $x = b$. At the single root, the curve crosses the x axis; at the double root, there is a point of tangency with the x axis. The leading coefficient is positive, so the graph is as below:



— Nota Bene —

In the vicinity of a single root, the curve looks like a polynomial of degree 1, i.e. a straight line. In the vicinity of a double root, the curve looks like a polynomial of degree 2, i.e. a parabola.

189. The probability that the first name drawn is the first name alphabetically is $1/10$. The probability that the second name drawn is the second name alphabetically is then $1/9$. For the third, $1/8$. So, the probability is $p = 1/10 \times 1/9 \times 1/8 = 1/720$.

190. Since the vectors, added tip-to-tail, form a closed perimeter, their sum must be zero. This gives

$$\begin{aligned} &\mathbf{a} + \mathbf{b} + \mathbf{c} - \mathbf{a} + \mathbf{d} = 0 \\ \Longrightarrow\ &\mathbf{b} + \mathbf{c} + \mathbf{d} = 0, \text{ as required.} \end{aligned}$$

191. In each case, the exponential grow without limit, so the $+1$ in the denominator becomes negligible. So, the limits are

(a) ∞, as the function approaches $\left(\frac{3}{2}\right)^x$,
(b) 1, as the function approaches $\left(\frac{3}{3}\right)^x$,
(c) 0, as the function approaches $\left(\frac{3}{4}\right)^x$.

Nota Bene

In mathematics, a limit is

the value towards which an expression is heading.

So, the limit of the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is zero. The definition in italics doesn't require attainment of the limit; indeed, many limits are not attained.

192. The equations for equilibrium are:

$$\begin{aligned} \updownarrow &: 3P - (P + 8) = 0 \\ \leftrightarrow &: Q + 20 - 8P = 0. \end{aligned}$$

The first equation gives $P = 4$, then $Q = 12$.

Nota Bene

The answers should not have units of Newtons. The numbers 20 and 8 in the diagram do not have units of Newtons, meaning that P and Q, which are added to those values, must also be unitless.

193. The simultaneous equations are

$$\begin{aligned} x + y &= 4\sqrt{2}, \\ xy &= 6. \end{aligned}$$

Substituting for y gives

$$\begin{aligned} &x\big(4\sqrt{2} - x\big) = 6 \\ \Longrightarrow\ &x^2 - 4\sqrt{2}x + 6 = 0. \end{aligned}$$

Using the quadratic formula,

$$\begin{aligned} x &= \frac{4\sqrt{2} \pm \sqrt{32 - 24}}{2} \\ &= 2\sqrt{2} \pm \sqrt{2} \\ &= \sqrt{2}, 3\sqrt{2}. \end{aligned}$$

So, the dimensions x and y of the rectangle are $\sqrt{2}$ and $3\sqrt{2}$. Using Pythagoras, the diagonals are $\sqrt{18 + 2} = \sqrt{20}$ cm long.

194. (a) By independence, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$. So, $\mathbb{P}(A) \times \mathbb{P}(B) = \frac{1}{12}$, which gives $\mathbb{P}(B) = \frac{1}{3}$.

(b) For any events A and B, the following formula (inclusion-exclusion) holds:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

This gives $\mathbb{P}(A \cup B) = \frac{1}{2}$.

195. Solving simultaneously,

$$\begin{aligned} &x^2 = 2x - 1 \\ \Longleftrightarrow\ &(x - 1)^2 = 0. \end{aligned}$$

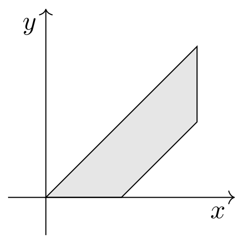
Since $(x-1)$ is a repeated factor, $x = 1$ is a double root. Hence, the line must be tangent to the curve at this point.

Nota Bene

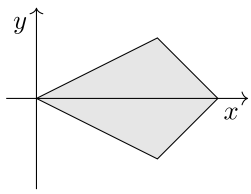
Why does finding a double root in the equation for intersections guarantee a point of tangency?

Consider the equation $\mathrm{f}(x) - \mathrm{g}(x) = 0$, whose roots are intersections of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$. The quantity $\mathrm{f}(x) - \mathrm{g}(x)$ is the y difference between the curves. It has a sign associated with it. If its sign changes, then the curves must have crossed over. But a double root corresponds to a squared factor, which guarantees that there is no such sign change. So, a double root guarantees a point of tangency.

196. (a) The quadrilateral is an (isosceles) trapezium:



(b) The quadrilateral is a kite:



197. Integrating the derivative,

$$\begin{aligned} &\frac{dy}{dx} = 4x - 1 \\ \Longrightarrow\ &y = \int 4x - 1\, dx \\ &\quad = 2x^2 - x + c. \end{aligned}$$

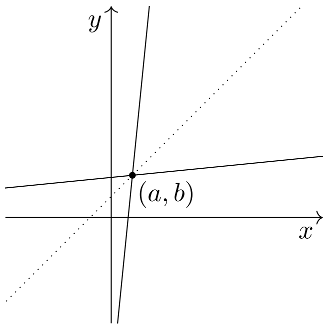
Substituting $(0, 3)$ gives $y = 2x^2 - x + 3$.

198. We multiply by $(1-x)$ to get

$$1 - x + px \equiv q.$$

Since this is an identity, we can equate coefficients. Those of x give $-1+p=0$, so that $p=1$. The constant terms give $q=1$.

199. These are straight lines through a general point (a,b), with reciprocal gradients 10 and $^1/_{10}$.



——— Nota Bene ———

Reciprocating the gradient turns $\frac{\Delta y}{\Delta x}$ into $\frac{\Delta x}{\Delta y}$. This is a switch of the roles of Δx and Δy, equivalent to reflection in a line with gradient 1 (dotted above).

200. The circumference of a unit circle is 2π. The length scale factor from an arc of unit length to the full circumference is, therefore, $\frac{2\pi}{1} = 2\pi$. This also scales the angle, so the circumference subtends an angle of $1 \times 2\pi = 2\pi$ radians. □

——— End of 2nd Hundred ———